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On the spacing distribution of the Riemann zeros: corrections to the asymptotic result

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Abstract
It has been conjectured that statistical properties of zeros of the Riemann zeta function near $z = 1/2 + \mathrm{i}E$ tend, as $E \to \infty$, to the distribution of eigenvalues of large random matrices from the unitary ensemble. At finite E, numerical results show that the nearest-neighbour spacing distribution presents deviations with respect to the conjectured asymptotic form. We give here arguments indicating that to leading order these deviations are the same as those of unitary random matrices of finite dimension $N_{\text{eff}} = \ln(E/2\pi)/\sqrt{12\beta}$, where $\beta = 1.573\,14\ldots$ is a well-defined arithmetic constant.

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1. Introduction

The study of connections between random matrix theory and properties of the Riemann zeta function, $\zeta(z)$, has recently known significant developments [1, 2]. A central point is Hugh Montgomery's (generalized) conjecture [3] that in the asymptotic limit (high on the critical line for $\zeta(1/2 + \mathrm{i}E)$), the fluctuation properties of non-trivial Riemann zeros are the same as for the circular unitary ensemble (CUE_N) of $N \times N$ random matrices (with Haar measure) in the limit of large dimensionality, $N \to \infty$. In particular, the normalized pair correlation function of the Riemann zeros with $E \to \infty$ is conjectured to be

$$R_2(s) = 1 - \left(\frac{\sin \pi s}{\pi s}\right)^2, \tag{1}$$

where s is the unfolded distance between zeros (i.e. the mean spacing is set to 1).

Nowadays, the theory of random matrices is used in number theory to obtain deep conjectures about the moments of zeta functions and the statistical properties of their zeros (see e.g. [4]). These conjectures are at present beyond the reach of exact methods and their validity and acceptance are largely based on large-scale numerical calculations.

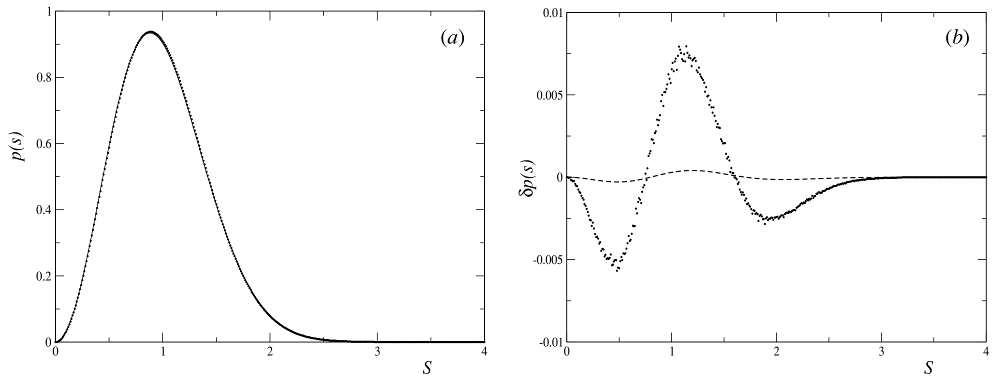

Figure 1. Nearest-neighbour spacing distribution of the Riemann zeros located in a window near $E = 2.504\,1178 \times 10^{15}$. (*a*) Numerical results (dots, from Odlyzko) compared to the asymptotic CUE curve (full line, almost indistinguishable). (*b*) Difference between the numerical result and the asymptotic CUE curve (dots) compared to the difference between the spacing distribution of CUE matrices of size $N_0 = 33.62$ (see equation (3)) and the asymptotic curve (dashed line).

A prominent example of this is the work of Andrew Odlyzko who, since the late 1970s, started accurate and extensive numerical computations of Riemann zeros [5]. One of his main results is that in the limit $E \to \infty$, correlation functions of Riemann zeros do agree with random matrix predictions. For instance, in figure 1(*a*) the density distribution $p(s)$ of spacings among consecutive zeros of $\zeta(1/2 + \mathrm{i}E)$ (called the nearest-neighbour spacing distribution in the random matrix literature) is plotted for a billion zeros around the 10^{16} th zero. The agreement with the asymptotic CUE prediction $p_0(s)$ (or GUE, which is asymptotically equivalent) is remarkable.

To study the approach to asymptotics, it is appropriate to look at the difference

$$\delta p(s) = p(s) - p_0(s) \tag{2}$$

between the computed and the conjectured distributions (see figure 1(*b*)). Though this difference is small (of the order of 10^{-2}), it has a clear structure with a nontrivial s dependence.

One may wonder how this difference compares with the one obtained within the CUE_N random matrix theory, namely the difference between the asymptotic and the finite N calculation. A natural assumption would be to choose $N = N_0$ where

$$N_0 = \ln\left(\frac{E}{2\pi}\right). \tag{3}$$

This matrix size is obtained by equating the local density of the zeros at height E along the critical line to the density of eigenvalues of the $N \times N$ unitary matrix (cf [1]). This dimensional correspondence has been successful when comparing statistical properties of $\zeta(1/2 + \mathrm{i}E)$ with those of characteristic polynomials of CUE_N matrices of size $N = N_0$ [1].

The Riemann zeros in figure 1 are located in a window around $E = 2.504\,1178 \times 10^{15}$, which gives $N_0 = 33.6188$. The difference between the finite N (with $N = N_0$) and the asymptotic nearest-neighbour spacing distribution is represented by a dashed line in figure 1(*b*) (in all numerical calculations the integer N is taken as the nearest integer to the theoretical estimate). Though the functional form of the correction is qualitatively correct, its amplitude is clearly too small (by a factor of order 20). In his paper [5], Odlyzko commented: ‘Clearly there is structure in the difference graph, and the challenge is to understand where it comes from'. The purpose of this note is to explain this difference by using the whole conjectured form of correlation functions of the Riemann zeros.

In section 2 the two-point correlation function of the Riemann zeros obtained in [6] is considered. By expanding it for small values of the argument, it is demonstrated that to leading order in the small parameter $1/\overline{\rho}$, where $\overline{\rho}$ is the average density of zeros, the correction term is the same as for CUE_N unitary random matrices with a well-defined value of $N = N_{\mathrm{eff}}$ (cf equation (18)). In section 3 we argue that the first sub-leading corrections in all correlation functions of the Riemann zeros are reduced to a change of the random matrix kernel only. These corrections happen to be the same as those of CUE_N matrices of size $N = N_{\mathrm{eff}}$. In particular, this leads to the conclusion that the leading correction to the nearest-neighbour spacing distribution of the Riemann zeros is the same as for CUE_N matrices with the cited value of N. We have checked numerically that this agrees well with Odlyzko's data. In the appendix the expansion of the three-point function of the Riemann zeros is worked out explicitly. It is shown that the dominant correction terms result again from the change of the kernel.

2. Two-point correlation function

A heuristic formula for the two-point correlation function of the Riemann zeros was obtained by Bogomolny and Keating in [6] using the Hardy–Littlewood conjecture of the distribution of prime pairs (for more details see [7, 8]). It states that the two-point function of Riemann zeros, $r_2(\epsilon)$, is the sum of three terms

$$r_2(\epsilon) = \overline{\rho}^2 + r_2^{(\mathrm{diag})}(\epsilon) + r_2^{(\mathrm{off})}(\epsilon), \tag{4}$$

where $\overline{\rho}$ is the smooth asymptotic density of zeros

$$\overline{\rho} = \frac{1}{2\pi} \ln\left(\frac{E}{2\pi}\right), \tag{5}$$

and the diagonal, $r_2^{(\mathrm{diag})}(\epsilon)$, and off-diagonal, $r_2^{(\mathrm{off})}(\epsilon)$, parts are given by the following convergent expressions:

$$r_2^{(\mathrm{diag})}(\epsilon) = -\frac{1}{4\pi^2}\frac{\partial^2}{\partial\epsilon^2}[\ln|\zeta(1+\mathrm{i}\epsilon)|^2] - \frac{1}{4\pi^2}\sum_p \left(\frac{\ln^2 p}{(p^{1+\mathrm{i}\epsilon}-1)^2} + \mathrm{c.c.}\right) \tag{6}$$

and

$$r_2^{(\mathrm{off})}(\epsilon) = \frac{1}{4\pi^2}|\zeta(1+\mathrm{i}\epsilon)|^2\,\mathrm{e}^{2\pi\,\mathrm{i}\overline{\rho}\epsilon}\prod_p\left[1 - \frac{(p^{\mathrm{i}\epsilon}-1)^2}{(p-1)^2}\right] + \mathrm{c.c.} \tag{7}$$

Here, the summation and product are taken over all primes p.

The unfolded two-point correlation function is obtained by measuring distances between zeros in units of the local mean spacing,

$$R_2(s) = \frac{1}{\overline{\rho}^2} r_2\left(\frac{s}{\overline{\rho}}\right). \tag{8}$$

In [8] it was checked numerically that these formulae agree very well with Odlyzko's results for the two-point correlation function of Riemann zeros.

We are interested in the corrections to the asymptotic behaviour of $R_2(s)$ in the limit when $E \to \infty$. In this limit $\overline{\rho} \to \infty$ and the argument of r_2 in equation (8) becomes small (keeping s finite). Therefore, one can expand $r_2(\epsilon)$ for $\epsilon \ll 1$. There are two types of terms in this expansion. The first comes from the convergent sum and product, and its computation is straightforward. The second corresponds to the expansion of $\zeta(1+x)$ when $|x| \to 0$ and can be obtained from the known series

$$\zeta(1+x) = \frac{1}{x} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n x^n,$$

where γ_n are the Stieljes constants

$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{\ln^n k}{k} - \frac{\ln^{n+1} m}{m+1} \right).$$

In particular, $\gamma_0 \approx 0.577\,216$, and $\gamma_1 \approx -0.072\,816$.

Collecting the first two terms in the expansion, one gets

$$r_2^{(\text{diag})}(\epsilon) = -\frac{1}{2\pi^2\epsilon^2} - \frac{\beta}{2\pi^2} + \mathcal{O}(\epsilon^2) \tag{9}$$

and

$$r_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} \left[\frac{1}{\epsilon^2} + \beta + \mathrm{i}\delta\epsilon + \mathcal{O}(\epsilon^2) \right] \mathrm{e}^{2\pi\,\mathrm{i}\bar{\rho}\epsilon} + \text{c.c.}, \tag{10}$$

where β and δ are real constants given by the following convergent sums:

$$\beta = \gamma_0^2 + 2\gamma_1 + \sum_p \frac{\ln^2 p}{(p-1)^2} \approx 1.573\,14 \tag{11}$$

and

$$\delta = \sum_p \frac{\ln^3 p}{(p-1)^2} \approx 2.3157. \tag{12}$$

The unfolded two-point correlation function takes, therefore, the form

$$R_2(s) = 1 - \frac{\sin^2(\pi s)}{\pi^2 s^2} - \frac{\beta}{\pi^2\bar{\rho}^2} \sin^2(\pi s) - \frac{\delta}{2\pi^2\bar{\rho}^3} s \sin(2\pi s) + \mathcal{O}(1/\bar{\rho}^4). \tag{13}$$

Equation (13) expresses the two-point correlation function of Riemann zeros as the asymptotic random matrix result given by equation (1) plus corrections proportional to inverse powers of the average density of zeros.

To interpret this result, it is convenient to analyse the corrections to the asymptotic two-point function (1) for the circular unitary ensemble (CUE_N) with finite N.

For N-dimensional CUE_N matrices, it is known that the unfolded two-point correlation function has the following form [9]:

$$R_2^{(\text{CUE}_N)}(s) = 1 - \left(\frac{\sin(\pi s)}{N \sin(\pi s/N)} \right)^2. \tag{14}$$

Expanding this expression in powers of $1/N$, one gets

$$R_2^{(\text{CUE}_N)}(s) = 1 - \frac{\sin^2(\pi s)}{\pi^2 s^2} - \frac{1}{3N^2} \sin^2(\pi s) - \frac{(\pi s)^2}{N^4} \sin^2(\pi s) + \mathcal{O}(1/N^6). \tag{15}$$

This formula expresses the correlation function as the asymptotic result, equation (1), plus corrections proportional to inverse even powers of the matrix dimension.

The comparison of equations (15) and (13) shows that the leading terms coincide, as conjectured by Montgomery. To relate sub-leading terms we proceed as follows. In fact, up to $\mathcal{O}(1/\overline{\rho}^4)$ in equation (13), the term of order $1/\overline{\rho}^3$ can be absorbed in the second term by rescaling the s variable

$$R_2(s) = 1 - \frac{\sin^2(\pi s)}{\pi^2 s^2} - \frac{\beta}{\pi^2 \overline{\rho}^2} \sin^2(\pi \alpha s) + \mathcal{O}(1/\overline{\rho}^4), \tag{16}$$

where

$$\alpha = 1 + \frac{\delta}{2\pi \overline{\rho} \beta} = 1 + \frac{C}{\ln(E/2\pi)}, \tag{17}$$

with $C = \delta/\beta \approx 1.4720$.

The comparison of equations (15) and (16) shows that

(i) To leading order in $1/\overline{\rho}$, the two-point correlation function of the Riemann zeros is the same as the one of eigenvalues of random CUE_N matrices of effective dimension $N = N_{\text{eff}}$, with

$$N_{\text{eff}} = \frac{\pi \overline{\rho}}{\sqrt{3\beta}} = \frac{\ln(E/2\pi)}{\sqrt{12\beta}}, \tag{18}$$

(ii) the next-to-leading order is obtained by rescaling the variable s in the first correction term according to

$$s \to \alpha s. \tag{19}$$

Here, the values of β and α are defined by equations (11) and (17) respectively.

The effective matrix size obtained from our analysis is, therefore, different from N_0 as given by equation (3), the relation between them being a constant multiplicative arithmetic factor $N_{\text{eff}} = (12\beta)^{-1/2} N_0 \approx 0.230\,158 \ln(E/2\pi)$.

For the two-point function, this result is just a reformulation of the first terms of the expansion of equations (6) and (7). In the following section, we argue that it applies to all correlation functions of the Riemann zeros and, in particular, to the nearest-neighbour spacing distribution.

3. The nearest-neighbour spacing distribution

In general, to compute the nearest-neighbour spacing distribution, it is necessary to know correlation functions with an arbitrary number of points. Though for the Riemann zeros there exist heuristic methods which permit, in principle, to obtain all correlation functions [2, 7, 10], the computations are cumbersome (cf the appendix) and we follow here another path.

It is well known [9] that, within random matrix theory, correlation functions of unitary ensembles are given by the determinant of a certain kernel K:

$$R_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))|_{i,j=1,\ldots,n}. \tag{20}$$

For the standard unitary ensemble (GUE) of random matrices in the universal limit, the kernel is [9]

$$K_0(s) = \frac{\sin(\pi s)}{\pi s}. \tag{21}$$

It is this kernel which leads to all universal random matrix predictions in the bulk of the spectrum.

For other ensembles the kernel may be different. In particular, for CUE_N of random unitary matrices, the kernel has the form [9]

$$K_N(x, y) = \frac{\sin(\pi(x - y))}{N \sin(\pi(x - y)/N)}. \tag{22}$$

When $N \to \infty$ this kernel can be expanded in inverse powers of N

$$K_N(x, y) = K_0(x-y) + \frac{1}{N^2} K_1(x-y) + \mathcal{O}(1/N^4), \tag{23}$$

with the first correction, $K_1(s)$, being

$$K_1(s) = \frac{\pi s}{6} \sin(\pi s). \tag{24}$$

Some physical arguments [11] indicate that to leading order and under quite general conditions, deviations from standard random matrix theory in systems with no time reversal invariance are reduced to a change of the kernel only. The correction term is the same as for CUE_N matrices with a certain effective matrix size which plays the role of the expansion parameter.

From equation (13), it follows that the two-point function of the Riemann zeros can be rewritten in the following form:

$$R_2(s) = 1 - \left(\frac{\sin \pi s}{\pi s} + s \frac{\beta}{2\pi \overline{\rho}^2} \sin(\pi s) + s^2 \frac{\delta}{2\pi \overline{\rho}^3} \cos(\pi s) + \mathcal{O}(1/\overline{\rho}^4) \right)^2, \tag{25}$$

from which one finds that up to order $1/\overline{\rho}^4$ this function can be written in the determinantal form

$$R_2(s) = \begin{vmatrix} 1 & K(s) \\ K(s) & 1 \end{vmatrix}, \tag{26}$$

with the kernel

$$K(s) = K_0(s) + k_1(s). \tag{27}$$

Here, $K_0(s)$ is the universal kernel (21) and $k_1(s)$ is the effective correction to the universal result which, neglecting terms of order $1/\overline{\rho}^4$, reads

$$k_1(s) = s \frac{\beta}{2\pi \overline{\rho}^2} \sin(\pi s) + s^2 \frac{\delta}{2\pi \overline{\rho}^3} \cos(\pi s) = \frac{\pi s}{6 N_{\mathrm{eff}}^2} \sin(\pi \alpha s), \tag{28}$$

where N_{eff} and α are the same as in equations (18) and (17) respectively. As expected, the dominant correction has the form as for CUE_N matrices with $N = N_{\mathrm{eff}}$.

In the appendix, we demonstrate that leading correction terms in the expansion of the three-point correlation function of the Riemann zeros result from exactly the same change of the kernel. Therefore, we conjecture that, to leading order, all correlation functions of the Riemann zeros are the same as those of CUE_N matrices with effective dimension given by equation (18). In particular, it means that the nearest-neighbour spacing distribution of the Riemann zeros can be calculated as follows.

First, find the expansion of the nearest-neighbour spacing distribution for CUE_N random unitary matrices in inverse powers of N, namely[3]

$$p^{(\mathrm{CUE}_N)}(s) = p_0(s) + \frac{1}{N^2} p_1^{(\mathrm{CUE})}(s) + \mathcal{O}(1/N^4). \tag{29}$$

Second, to leading order, at a given height E on the critical line, replace in this equation N by N_{eff} given by equation (18). Finally, approximate the next-to-leading term by rule (ii) (cf equation (19)).

In such approximation, the nearest-neighbour distribution for the Riemann zeros equals the universal random matrix result, $p_0(s)$, plus the correction terms $\delta p(s)$ where

$$\delta p(s) = \frac{1}{N_{\mathrm{eff}}^2} p_1^{(\mathrm{CUE})}(\alpha s) + \mathcal{O}\left(1 \big/ N_{\mathrm{eff}}^4\right). \tag{30}$$

[3] Note that the corresponding expansion for GUE_N is different and includes, in particular, a term proportional to $(-1)^N/N$ (cf [12], in particular the end of sections V.A and V.D)

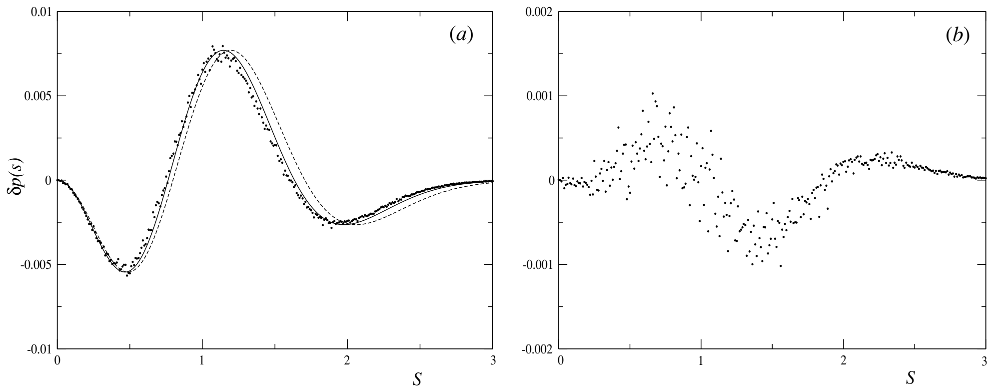

Figure 2. (*a*) Difference between the nearest-neighbour spacing distribution of the Riemann zeros and the asymptotic CUE distribution for a billion zeros located in a window near $E = 2.504\,1178 \times 10^{15}$ (dots), compared to the theoretical prediction equation (30) (full line). The dashed line does not include the rescaling of s. (*b*) Difference between the numerical Riemann values (dots) and the full curve (theory) of part (*a*).

The expansion (29) is difficult to treat analytically. To determine $p_1^{(\mathrm{CUE})}(s)$, we rather use the following numerical method.

For CUE_N, the nearest-neighbour spacing distribution may be expressed as the second derivative [9] of $E(s)$:

$$p^{(\mathrm{CUE}_N)}(s) = \frac{\mathrm{d}^2 E(s)}{\mathrm{d}s^2}, \tag{31}$$

where $E(s)$ is the probability of finding a hole of size s in the spectrum. It is given by the following determinant [9]:

$$E(s) = \det\left[\delta_{jk} - \frac{\sin(\pi s(j-k)/N)}{\pi(j-k)}\right], \qquad 1 \leqslant j, \quad k \leqslant N. \tag{32}$$

This determinant can be calculated numerically and the correction term is then obtained by computing $p_1^{(\mathrm{CUE})}(s) = N^2[p^{(\mathrm{CUE}_N)}(s) - p_0(s)]$ for increasing values of N. This procedure is quite robust and a good convergence is found for N of order 40. A more refined method would be to use a nonlinear differential equation for $p^{(\mathrm{CUE}_N)}(s)$ as derived in [12].

Figure 2(*a*) shows the comparison between the numerical results and equation (30) for zeros located on a window around $E = 2.504\,1178 \times 10^{15}$ (as in figure 1). The effective matrix size is $N_{\mathrm{eff}} = 7.7376$ (instead of $N_0 = 33.6188$), and $\alpha = 1.0438$. The agreement is quite good and shows that N_{eff} is the correct effective matrix size in this case. For comparison, we have plotted as a dashed curve the theoretical formula (30) without the rescaling of the s variable.

Figure 2(*b*) is a plot of the difference between Odlyzko's results and the prediction (30). There is still some structure visible, which might be attributed to the $\mathcal{O}(N_{\mathrm{eff}}^{-4})$ correction. To test the convergence, we have made the same plot but now using one billion zeros located on a window around $E = 1.306\,643\,44 \times 10^{22}$, which corresponds to $N_{\mathrm{eff}} = 11.2976$ (instead of $N_0 = 49.0864$) and $\alpha = 1.0300$ (figure 3(*a*)). Now the agreement is clearly improved. The difference between the prediction (30) and the numerical results, plotted in figure 3(b), displays a structureless remain.

These results clearly demonstrate that heuristic formulae for statistical distributions of the Riemann zeros permit to explain very well even tiny details of large-scale numerical

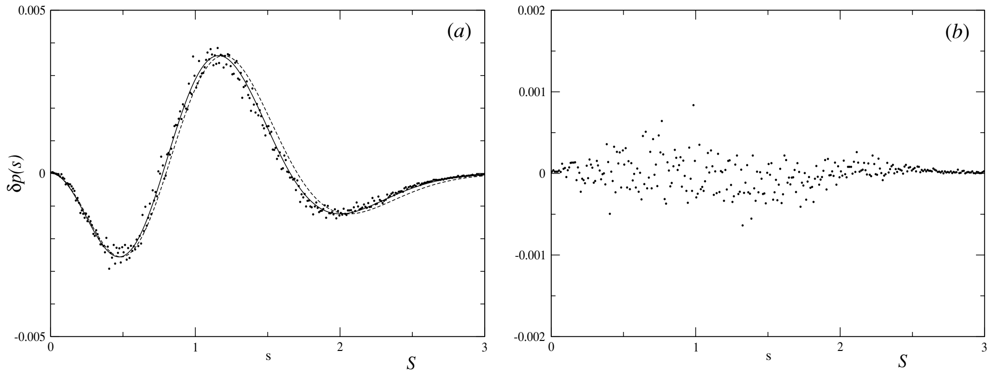

Figure 3. Same as in figure 2 but for a billion zeros located in a window near $E = 1.306\,643\,44 \times 10^{22}$.

calculations of these zeros. This agreement gives additional support to the conjectures on which these formulae are based.

4. Conclusion

Using heuristic arguments, we derive a formula which includes corrections to the nearest-neighbour spacing distribution of the Riemann zeta function zeros. We argue that to leading order the correction is the same as for random matrices from the circular unitary ensemble of size $N_{\text{eff}} \approx 0.230\,158 \ln(E/2\pi)$. We propose to describe the next-to-leading order correction as a rescaling of the dominant term.

Two main conjectures have been used. The most important is the explicit expression for the two-point correlation function of the Riemann zeros (equations (6) and (7)) obtained in [6]. The second is a statement that the leading order deviations from random matrix predictions reduce to a change of the kernel. This has been checked for two- and three-point functions and presumably can be extended to n-point functions.

Let us finally mention that at a finite height on the critical line, the appropriate symmetry group of the Riemann zeta function is conjectured to be $USp(2N)$ [13]. The symmetry dependence of correction terms clearly deserves further investigation.

Acknowledgment

We are thankful to A Odlyzko for useful discussions as well as for providing us with unpublished data.

Appendix. Expansion of the three-point correlation function

Using a heuristic method proposed in [7], it can be demonstrated [14] that the three-point correlation function of the Riemann zeros has the following form:

$$r_3(e_1, e_2, e_3) = \overline{\rho}^3 + \overline{\rho} r_2^{(c)}(e_{12}) + \overline{\rho} r_2^{(c)}(e_{23}) + \overline{\rho} r_2^{(c)}(e_{31}) + r_3^{(c)}(e_1, e_2, e_3). \tag{A.1}$$

Here and below, the notation e_{ij} indicates the difference of e_i and e_j:

$$e_{ij} = e_i - e_j. \tag{A.2}$$

In the above expression, $r_3^{(c)}(e_1, e_2, e_3)$ is the connected part of the three-point function that is expressed as the sum of two terms:

$$r_3^{(c)}(e_1, e_2, e_3) = r_3^{(\mathrm{diag})}(e_1, e_2, e_3) + r_3^{(\mathrm{off})}(e_1, e_2, e_3), \tag{A.3}$$

where the diagonal part $r_3^{(\mathrm{diag})}(e_1, e_2, e_3)$ is given by the convergent sum over prime numbers:

$$\begin{aligned} r_3^{(\mathrm{diag})}(e_1, e_2, e_3) = -\frac{1}{(2\pi)^3} \sum_p \ln^3 p \Bigg(&\frac{1}{(p^{1-\mathrm{i}e_{12}} - 1)(p^{1-\mathrm{i}e_{13}} - 1)} \\ &+ \frac{1}{(p^{1-\mathrm{i}e_{21}} - 1)(p^{1-\mathrm{i}e_{23}} - 1)} + \frac{1}{(p^{1-\mathrm{i}e_{32}} - 1)(p^{1-\mathrm{i}e_{31}} - 1)} \Bigg) + \text{c.c.}, \end{aligned} \tag{A.4}$$

and the oscillating part $r_3^{(\mathrm{off})}(e_1, e_2, e_3)$ contains both convergent sums and products and formally divergent parts which can be expressed through values of the Riemann zeta function on the axis $\mathrm{Re}(s) = 1$:

$$\begin{aligned} r_3^{(\mathrm{off})}(e_1, e_2, e_3) = &-\frac{1}{(2\pi\,\mathrm{i})^3} \mathrm{e}^{2\pi\,\mathrm{i}\overline{\rho} e_{12}} |\zeta(1 + \mathrm{i}e_{12})|^2 \prod_p \left(1 - \frac{(1 - p^{\mathrm{i}e_{12}})^2}{(p-1)^2}\right) \\ &\times \Bigg[\frac{\partial}{\partial e_3} \ln \left| \frac{\zeta(1 + \mathrm{i}e_{32})}{\zeta(1 + \mathrm{i}e_{31})} \right|^2 - \mathrm{i} \sum_p \ln p \Bigg(\frac{p^{\mathrm{i}e_{12}} - 1}{(p^{1+\mathrm{i}e_{23}} - 1)(p^{1+\mathrm{i}e_{13}} - 1)} \\ &+ \frac{p^{\mathrm{i}e_{12}} - 1}{(p^{1-\mathrm{i}e_{13}} - 1)(p^{1-\mathrm{i}e_{22}} - 1)} \\ &+ \frac{(1 - p^{\mathrm{i}e_{12}})^2}{p - 2 + p^{\mathrm{i}e_{12}}} \left(\frac{1}{p^{1-\mathrm{i}e_{31}} - 1} + \frac{1}{p^{1-\mathrm{i}e_{23}} - 1} \right) \Bigg) \Bigg] + \text{permutations}. \end{aligned} \tag{A.5}$$

In the last expression, only one term proportional to $\mathrm{e}^{2\pi\,\mathrm{i}\overline{\rho} e_{12}}$ is written explicitly. The whole expression contains two other terms proportional to $\mathrm{e}^{2\pi\,\mathrm{i}\overline{\rho} e_{23}}$ and $\mathrm{e}^{2\pi\,\mathrm{i}\overline{\rho} e_{31}}$ corresponding to the cyclic permutations of three indices 1, 2 and 3 plus the complex conjugation of the result. Equations (A.4) and (A.5) may also be obtained from the ratio conjecture [15].

The unfolding of the three-point function (the analogue of equation (8)) corresponds to the following scaling transformation:

$$R_3(s_1, s_2, s_3) = \frac{1}{\overline{\rho}^3} r_3 \left(\frac{s_1}{\overline{\rho}}, \frac{s_2}{\overline{\rho}}, \frac{s_3}{\overline{\rho}} \right). \tag{A.6}$$

From equation (A.4), it follows that the diagonal part of the three-point correlation function with $e_{ij} = s_{ij}/\overline{\rho}$ where $s_{ij} \equiv s_i - s_j$ is fixed and $\overline{\rho} \to \infty$ to leading order is

$$R_3^{(\mathrm{diag})}(s_1, s_2, s_3) = -\frac{6}{(2\pi\overline{\rho}^3)^3} \sum_p \frac{\ln^3 p}{(p-1)^2} = -\frac{3}{4\pi^3}\delta', \tag{A.7}$$

where

$$\delta' = \frac{\delta}{\overline{\rho}^3}, \tag{A.8}$$

with δ as in equation (12).

For the oscillatory part (A.5) in the same limit, one has

$$\begin{aligned} R_3^{(\mathrm{off})}(s_1, s_2, s_3) = &-\frac{1}{(2\pi\,\mathrm{i})^3} \mathrm{e}^{2\pi\,\mathrm{i} s_{12}} \left(\frac{1}{s_{12}^2} + \beta' + \mathrm{i}\delta' s_{12} \right) \\ &\times \left[\frac{\partial}{\partial s_3} \left(-2 \ln s_{32} + 2 \ln s_{31} + \frac{\gamma_0^2 + 2\gamma_1}{\overline{\rho}^2} (s_{32}^2 - s_{31}^2) \right) \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{2s_{12}}{\bar\rho^2}\sum_p \frac{\ln^2 p}{(p-1)^2} + \frac{3\mathrm{i}s_{12}^2}{\bar\rho^3}\sum_p \frac{\ln^3 p}{(p-1)^2}\Bigg]\\
&= -\frac{1}{(2\pi\,\mathrm{i})^3}\,\mathrm{e}^{2\pi\,\mathrm{i}s_{12}}\left(\frac{1}{s_{12}^2}+\beta'+\mathrm{i}\delta' s_{12}\right)\left[-2\frac{s_{12}}{s_{23}s_{31}}+2s_{12}\beta'+3\mathrm{i}\delta' s_{12}^2\right]\\
&= \frac{1}{(2\pi\,\mathrm{i})^3}\,\mathrm{e}^{2\pi\,\mathrm{i}s_{12}}\left(\frac{1}{s_{12}s_{23}s_{31}} - 2\beta'\left(\frac{1}{s_{12}}+\frac{1}{s_{23}}+\frac{1}{s_{31}}\right)+\mathrm{i}\delta'\left(2\frac{s_{12}^2}{s_{23}s_{31}}-3\right)\right)\\
&+\text{permutations}\,. \qquad\qquad (\text{A.9})
\end{aligned}
$$

Here,

$$
\beta' = \frac{\beta}{\bar\rho^2}. \qquad (\text{A.10})
$$

To check that this result corresponds to the change of the kernel (28), one needs to calculate the first correction to the determinantal formula

$$
R_3(s_1,s_2,s_3) = \begin{vmatrix} 1 & K_{12} & K_{13}\\ K_{21} & 1 & K_{23}\\ K_{31} & K_{32} & 1\end{vmatrix}, \qquad (\text{A.11})
$$

where $K_{ij} = K_0(s_{ij}) + k_1(s_{ij})$ (Note that $K(0) = 1$). For the calculations below, it is convenient to rewrite the correction term (28) as follows:

$$
k_1(s) = \frac{1}{2\pi\mathrm{i}}\left(a(s)\,\mathrm{e}^{2\pi\,\mathrm{i}s} - a^*(s)\,\mathrm{e}^{-2\pi\,\mathrm{i}s}\right), \qquad (\text{A.12})
$$

with

$$
a(s) = \tfrac{1}{2}(\beta' s + \mathrm{i}\delta' s^2). \qquad (\text{A.13})
$$

Expanding the determinant and taking into account the correction term k_1 only to first order, one gets

$$
\begin{aligned}
R_3(s_1,s_2,s_3) &= 1 - (K_{12}K_{21} + K_{23}K_{32} + K_{31}K_{13}) + K_{12}K_{23}K_{31} + K_{21}K_{13}K_{32}\\
&= R_3^{(0)}(s_1,s_2,s_3) + r_3^{(1)}(s_1,s_2,s_3), \qquad (\text{A.14})
\end{aligned}
$$

where $R_3^{(0)}(s_1,s_2,s_3)$ is the determinant (A.11) computed using the standard kernel (21) and $r_3^{(1)}(s_1,s_2,s_3)$ is the first correction:

$$
\begin{aligned}
r_3^{(1)}(s_1,s_2,s_3) &= -2\big(K_{12}^{(0)}k_{21}^{(1)} + K_{23}^{(0)}k_{32}^{(1)} + K_{31}^{(0)}k_{13}^{(1)}\big)\\
&\quad + 2\big(K_{12}^{(0)}K_{23}^{(0)}k_{31}^{(1)} + K_{12}^{(0)}k_{23}^{(1)}K_{31}^{(0)} + k_{12}^{(1)}K_{23}^{(0)}K_{31}^{(0)}\big). \qquad (\text{A.15})
\end{aligned}
$$

Here, $K_{ij}^{(0)} = K_0(s_{ij})$ and $k_{ij}^{(1)} = k_1(s_{ij})$. The first three terms in this sum are corrections to the two-point correlation function. It has been checked in the previous section that they take the correct form. Only the last three terms are non-trivial.

The following calculations are straightforward:

$$
\begin{aligned}
k_{12}^{(1)}K_{23}^{(0)}K_{31}^{(0)} &= \frac{1}{(2\pi\mathrm{i})^3 s_{23}s_{31}}\left(a_{12}\,\mathrm{e}^{2\pi\mathrm{i}s_{12}} - a_{12}^*\,\mathrm{e}^{-2\pi\mathrm{i}s_{12}}\right)\left(\mathrm{e}^{2\pi\mathrm{i}s_{23}} - \mathrm{e}^{-2\pi\mathrm{i}s_{23}}\right)\left(\mathrm{e}^{2\pi\mathrm{i}s_{31}} - \mathrm{e}^{-2\pi\mathrm{i}s_{31}}\right)\\
&= \frac{1}{(2\pi\mathrm{i})^3 s_{23}s_{31}}\left[a_{12} - a_{12}^* + a_{12}\left(\mathrm{e}^{2\pi\mathrm{i}s_{12}} - \mathrm{e}^{-2\pi\mathrm{i}s_{23}} - \mathrm{e}^{-2\pi\mathrm{i}s_{31}}\right)\right.\\
&\quad\left. - a_{12}^*\left(\mathrm{e}^{-2\pi\mathrm{i}s_{12}} - \mathrm{e}^{2\pi\mathrm{i}s_{23}} - \mathrm{e}^{2\pi\mathrm{i}s_{31}}\right)\right].
\end{aligned}
$$

Here, $a_{ij} = a(s_{ij})$ where $a(s)$ is given by equation (A.13).

Computing the two other terms and combining similar exponents, we obtain that the connected part of the corrections has the form

$$r_3^{(1)}(s_1,s_2,s_3) = \frac{2}{(2\pi\mathrm{i})^3}\left(\frac{a_{12}-a_{12}^*}{s_{23}s_{31}}+\frac{a_{23}-a_{23}^*}{s_{12}s_{31}}+\frac{a_{31}-a_{31}^*}{s_{12}s_{23}}\right) + \frac{2}{(2\pi\mathrm{i})^3}\mathrm{e}^{2\pi\mathrm{i}s_{12}}\left(\frac{a_{12}}{s_{23}s_{31}}+\frac{a_{23}^*}{s_{12}s_{31}}+\frac{a_{31}^*}{s_{12}s_{23}}\right) + \text{ permutations.}$$

Using equation (A.13), one concludes that to first order in k_1 the three-point function changes as follows:

$$r_3^{(\mathrm{diag})}(s_1,s_2,s_3) = -\frac{\delta'}{4\pi^3}\left(\frac{s_{12}^2}{s_{23}s_{31}}+\frac{s_{23}^2}{s_{12}s_{31}}+\frac{s_{31}^2}{s_{12}s_{23}}\right) = -\frac{\delta'}{4\pi^3 s_{12}s_{23}e_{31}}\left(s_{12}^3+s_{23}^3+s_{31}^3\right)$$

and

$$r_3^{(\mathrm{off})}(s_1,s_2,s_3) = \frac{1}{(2\pi\mathrm{i})^3}\mathrm{e}^{2\pi\mathrm{i}s_{12}}\left[\beta'\left(\frac{s_{12}}{s_{23}e_{31}}+\frac{s_{23}}{s_{12}s_{31}}+\frac{s_{31}}{s_{12}s_{23}}\right) + \mathrm{i}\delta'\left(\frac{s_{12}^2}{s_{23}s_{31}}-\frac{s_{23}^2}{s_{12}s_{31}}-\frac{s_{31}^2}{s_{12}s_{23}}\right)\right] + \text{ permutations.}$$

By definition, one has $s_{12}+s_{23}+s_{31}=0$. Therefore,

$$s_{31}^3 = -(s_{12}+s_{23})^3 = -s_{12}^3 - s_{23}^3 - 3s_{12}^2s_{23} - 3s_{12}s_{23}^2. \tag{A.16}$$

Consequently,

$$s_{12}^3+s_{23}^3+s_{31}^3 = -3(s_{12}+s_{23})s_{12}s_{23} = 3s_{12}s_{23}s_{31}. \tag{A.17}$$

From these relations, one finally gets

$$r_3^{(\mathrm{diag})}(s_1,s_2,s_3) = -\frac{3}{4\pi^3}\delta' \tag{A.18}$$

and

$$r_3^{(\mathrm{off})}(s_1,s_2,s_3) = \frac{1}{(2\pi\mathrm{i})^3}\mathrm{e}^{2\pi\mathrm{i}s_{12}}\left[-2\beta'\left(\frac{1}{s_{12}}+\frac{1}{s_{23}}+\frac{1}{s_{31}}\right)+\mathrm{i}\delta'\left(2\frac{s_{12}^2}{s_{23}s_{31}}-3\right)\right] + \text{permutations}\,. \tag{A.19}$$

These equations coincide exactly with equations (A.7) and (A.9). This proves that the three-point correlation function of the Riemann zeros has, up to order $1/\bar{\rho}^3$, the determinantal form (A.11) as for standard random matrix ensembles but with the modified kernel as in equations (27) and (28).

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